4 Familiar results of the four Maxwell's equations

4.1 Eq. 1, a.k.a. Gauss's law

Starting from the first equation (161), we can derive some equations you already know: the electric field from a point charge, and the electric field between two parallel plates.

First, we integrate Eq. 161 over some unspecified volume V

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{65}$$

$$\int_V \nabla \cdot \mathbf{E} \quad \partial\tau = \int_V \frac{\rho}{\epsilon_0} \quad \partial\tau \tag{66}$$

where $\partial\tau$ is the volume element. We can use now a useful identity, the *divergence theorem*, or *Gauss's theorem*,

$$\boxed{\int_V \nabla \cdot \mathbf{v} \quad d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}} \tag{67}$$

where the closed integral is over the *surface* of the integration volume, taking the component of $\mathbf{v}$ normal to the surface.

If we think about $\mathbf{v}$ representing the direction of fluid flow, the divergence theorem can be rationalized by considering that the total spreading of the fluid over some volume (integrated divergence) has to leak out over the volume's boundary–it has nowhere else to go. In the context of the electric field, application of Gauss's theorem gives

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V \frac{\rho}{\epsilon_0} \quad \partial\tau \tag{68}$$

Point charge For the case of the point charge, we can take the integration volume as a sphere with radius R. We know ahead of time that the field from the point charge points away from the charge, so $\mathbf{E} \cdot d\mathbf{a} = E_r \quad da$: *the field is always constant, and normal to the boundary.* This fact makes the divergence theorem useful.

Taking the volume around the charge to be a sphere with radius R,

$$E(4\pi R^2) = \frac{Q}{\epsilon_0} \tag{69}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}} \tag{70}$$

If we are thinking about the force on an electron around this point charge, application of the Lorenz force yields

$$\mathbf{F} = e\mathbf{E} \tag{71}$$

$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}} \tag{72}$$

telling us, as we know, that two like charges are repelled.

Parallel plate capacitors Consider instead that we have two parallel plates, separated by a distance d, charged with charge Q. We know again that the electric field is constant between the plates, so we choose an appropriate volume (the "gaussian pillbox") to take advantage of this fact. Application of Equation 68 now gives

$$EA = \frac{Q}{\epsilon_0} \tag{73}$$

$$\frac{V}{d}A = \frac{Q}{\epsilon_0} \tag{74}$$

$$\frac{C}{A} = \frac{Q}{VA} = \frac{\epsilon_0}{d} \tag{75}$$

the capacitance per unit area of parallel plates.

4.2 Eq. 2: no monopoles

A brief look at the second Maxwell equation (eq. 162),

$$\nabla \cdot \mathbf{B} = 0 \tag{76}$$

points out a major difference between electric and magnetic fields. Magnetic fields B are divergenceless, implying that there is no such thing as magnetic "charge." As far as we know, there are **no magnetic monopoles**; every magnetic field line forms a closed loop.

What about "free poles?" People in the magnetic recording industry often mention *magnetic charge* as if it were a real entity, and in a certain sense, it is: a magnetized object serves as a source for a magnetic field at its boundary; this is the basis for magnetic recording. Why is there not "magnetic charge," like static electric charge?

This is getting a bit ahead of ourselves, but *inside materials*, rather than in vacuum, we have

$$\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H}) \tag{77}$$

where $\mathbf{M}$ is the *magnetization* of the material, and $\mathbf{H}$ is the magnetic field in vacuum, in the absence of $\mathbf{M}$. $\mathbf{H}$ is the field one can control using a wireloop. Applying the divergence to both sides tells us

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \tag{78}$$

which looks like $\nabla \cdot \mathbf{H} = \frac{1}{\mu_0}\rho_m$ if we can identify $\rho_m = -\mu_0 \nabla \cdot \mathbf{M}$. "Magnetic charge" is present at the boundary of a uniformly magnetized object because $\mathbf{M}$ drops off to zero there. I will talk about this more in the context of magnetic recording.

4.3 Eq. 3: Faraday's law

The third Maxwell equation reads

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{79}$$

Let's consider a wire loop, area A, with a uniform magnetic field B orthogonal to it. Faraday found that some change in the magnetic flux $\Phi = BA$ *through* the loop generates an electromotive force (voltage) *around* the loop:

$$\varepsilon = -\left(\frac{\partial \Phi}{\partial t}\right) \tag{80}$$

It's not immediately clear how we get from one to the other. Enter *Stokes's theorem*, which plays the same role for curls as Gauss's theorem does for divergences. If we consider a *patch of a surface* P, which does not need to be closed or any particular shape, bounded by the closed line L, the theorem tells us that

$$\int_P (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_L \mathbf{v} \cdot d\mathbf{l} \tag{81}$$

This expression is not quite as easy to understand intuitively. In terms of the fluid flow analogy on a stream, it says that if we would like to total the "swirl" in an area, we can just follow the flow around the boundary.

Graphically, this makes some sense because the rotation of adjacent loops could tend to cancel each other out *except* on the boundary.

We can take the surface defined by the wire loop as the patch P to be treated. Integrating Eq 79 over the loop area gives us

$$\int_{loop} (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = -\int_{loop} \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{a} \tag{82}$$

$$\oint_{loopboundary} \mathbf{E} \cdot d\mathbf{l} = -\left(\frac{\partial \Phi}{\partial t}\right) \tag{83}$$

$$\varepsilon = -\left(\frac{\partial \Phi}{\partial t}\right) \tag{84}$$

since the integrated electric field around the loop (V cm^{-1}), integrated over cm, gives us the voltage drop around the loop.

4.4 Eq. 4: Ampere's law

Finally we can examine the fourth Maxwell equation,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{85}$$

which in the steady state becomes

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{86}$$

application of the Stokes theorem, as carried out before, gives us

$$\oint_L \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_P \mathbf{J} \cdot d\mathbf{a} \tag{87}$$

Example: the solenoid This equation predicts the magnetic field inside a *solenoid*, a (hollow) tube wound circumferentially with wire. As with the previous examples, equation 88 becomes useful where geometry simplifies the problem. We choose a rectangular surface patch (or *Amperean loop*) with plane including the radius of the tube, so that the current will be normal to the patch surface.

We can then rewrite the right hand side as

$$\oint_L \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \tag{88}$$

where I_{enc} is the current enclosed by the patch boundary.

If the loop is drawn far enough away from the ends of the solenoid, $\mathbf{B}$ will point only along the solenoid axis. $\mathbf{B} \cdot d\mathbf{l}$ will be zero everywhere except within the loop: outside, the field is zero[2], and along the sides, $\mathbf{B}$ and $\mathbf{l}$ are orthogonal. Over the patch area, $I_{enc} = Ni$, where N is the number of turns along length s and i is the current through the solenoid coils. We then have

$$B = \mu_0 n i \tag{89}$$

where n is the number of turns per unit length. μ_0 [=] N A^{-2}; n [=] m^{-1}; i [=] A. The magnetic field B thus has units of N A^{-1} m^{-1}, called *Tesla* (T).

[2]To verify this, draw the loop outside of the solenoid; since no field is enclosed, the top and bottom contributions must be equal, and since the field goes to zero far away from the solenoid, equal must be zero

5 EM waves: "let there be light!"

Maxwell's equations predict that E and B *waves* can propagate, and EM energy or information can be transported through materials or vacuum. We experience these electromagnetic (EM) waves as light, radio waves, x-rays, cosmic rays.

Before exploring EM waves we should review some basics of propagating monochromatic waves. These features will be common to EM waves, sound waves, or quantum-mechanical matter waves.

5.1 General properties of waves

5.1.1 Basic description

A wave is a disturbance of a continuous medium which retains *a fixed shape* and moves with a *constant velocity.* In the case of ocean waves or sound waves, the disturbance represents the displacement of particles (water or air) from their rest positions. If we describe the magnitude of the disturbance by f, moving along direction z over time t, the behavior can be expressed mathematically as:

$$f(z,t) = f(z-vt,0) \tag{90}$$

$$= g(z-vt) \tag{91}$$

which states that any value of f, taken at point z, can be found at $z+vt$ at a time t later. So, the wave can be seen to move with velocity v.

A couple of examples of wave disturbances are

$$f(z,t) = \cos(z-vt+\delta) \tag{92}$$

$$f(z,t) = \frac{A}{b(z-vt)^2+1} \tag{93}$$

The wave does not need to be sinusoidal or periodic; the second example is a peak in f, Lorentzian in form, moving with velocity v.

5.1.2 The wave equation

All waves, that is, functions of the form $f(z,t) = g(z-vt)$, are solutions of the **wave equation:**

$$\left(\frac{\partial^2 f}{\partial z^2}\right) = \frac{1}{v^2}\left(\frac{\partial^2 f}{\partial t^2}\right) \tag{94}$$

This can be verified through the chain rule. Introducing $u = z - vt$,

$$\left(\frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial z}\right) \tag{95}$$

$$\left(\frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial u}\right) \tag{96}$$

$$\left(\frac{\partial^2 f}{\partial z^2}\right) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial u}\right) \tag{97}$$

$$= \left(\frac{\partial^2 f}{\partial u^2}\right)\left(\frac{\partial u}{\partial z}\right) \tag{98}$$

$$\tag{99}$$

so on the LHS we have

$$\left(\frac{\partial^2 f}{\partial z^2}\right) = \left(\frac{\partial^2 f}{\partial u^2}\right) \tag{100}$$

and similarly on the RHS we have

$$\left(\frac{\partial^2 f}{\partial t^2}\right) = v^2\left(\frac{\partial^2 f}{\partial u^2}\right) \tag{101}$$

5.1.3 Sinusoidal waves

A sinusoidal wave can be represented in its most general form as

$$f(z,t) = A\cos(kz - \omega t + \delta) \tag{102}$$

where k is the *wavenumber*, ω is the *angular frequency*, and δ is the *phase angle*.

Important quantities The wavenumber k is a measure of the spatial frequency of the wave; the wave repeats at positions given by the wavelength λ, where

$$k(n\lambda) = n2\pi \tag{103}$$

Thus the wavenumber is given by

$$k = \frac{2\pi}{\lambda} \tag{104}$$

The temporal frequency is given by ω. If the period is τ,

$$\omega(n\tau) = n2\pi \tag{105}$$

and the angular frequency is given by

$$\omega = \frac{2\pi}{\tau} \tag{106}$$

so named since it changes by 2π radians over a period; the frequency ν is instead

$$\nu = 1/\tau \tag{107}$$

Wave velocity How quickly does the wave move? We have the argument of the cosine as $kz - \omega t + \delta$; this remains constant for $z = \frac{\omega}{k}t$. Thus the velocity of the sinusoidal wave, the *phase velocity* v_p, is given by

$$v_p = \frac{\omega}{k} = \frac{\lambda}{\tau} \tag{108}$$

Complex representation Taking advantage of Euler's formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{109}$$

we can rewrite equation 102 as

$$f(z,t) = Re\left[Ae^{i(kz-\omega t+\delta)}\right] \tag{110}$$

and introducing the complex prefactor $\widetilde{A}$,

$$\widetilde{A} = Ae^{i\delta} \tag{111}$$

which subsumes the phase angle, we can represent the sinusoid in terms of the complex wave $\widetilde{f}(z,t)$

$$\widetilde{f}(z,t) = \widetilde{A}e^{i(kz-\omega t)} \tag{112}$$

of which we only need to take the real part

$$f(z,t) = Re\left[\widetilde{f}(z,t)\right] \tag{113}$$

to get back to the physical disturbance. We haven't introduced any new physical behavior in the complex representation; it is merely convenient for carrying out calculations.

5.1.4 Polarization

So far we have not said anything about the *direction* of the disturbance. In fact, the disturbance can be of two forms. In **longitudnal** waves, the disturbance is parallel to the direction of propagation, which we conventionally take to be along the $\hat{\mathbf{z}}$-direction. You could see longitudinal waves in a Slinky shaken on one end. These are also called *compression* waves, and are the form that sound waves take in air.

The form which is relevant for EM waves is **transverse**. For the transverse case, we could have displacement vertical...

$$\widetilde{\mathbf{f}}_v = \widetilde{A} e^{i(kz-\omega t)} \hat{\mathbf{x}} \tag{114}$$

or horizontal...

$$\widetilde{\mathbf{f}}_h = \widetilde{A} e^{i(kz-\omega t)} \hat{\mathbf{y}} \tag{115}$$

or in any unit direction $\hat{\mathbf{n}}$ which is perpendicular to the propagation direction.

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = 0 \tag{116}$$

and which can be given in terms of a *polarization angle* θ

$$\hat{\mathbf{n}} = \cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}} \tag{117}$$

as

$$\widetilde{\mathbf{f}}_h = \widetilde{A} e^{i(kz-\omega t)} \hat{\mathbf{y}} \tag{118}$$

5.2 Waves from Maxwell's equations

With the aid of a vector identity, it is relatively easy to show how Maxwell's equations lead to wave solutions in vacuum. Taking the curl of Maxwell no. 3, we have

$$\begin{aligned}\nabla \times (\nabla \times E) &= -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) && (119)\\ \nabla \times \nabla \times E &= -\frac{\partial}{\partial t}\nabla \times \mathbf{B} && (120)\\ &= -\mu_0\epsilon_0\left(\frac{\partial^2 \mathbf{E}}{\partial t^2}\right) && (121)\end{aligned}$$

where we used Maxwell no. 4, $\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$. To go further we need a vector calculus identity,

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v} \qquad (122)$$

which I won't attempt to motivate. This leaves us with

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0\epsilon_0\left(\frac{\partial^2 \mathbf{E}}{\partial t^2}\right) \qquad (123)$$

and using Maxwell no. 1,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \qquad (124)$$

where there is no charge ρ (in vacuum), we are left with

$$\boxed{\nabla^2\mathbf{E} = \mu_0\epsilon_0\left(\frac{\partial^2 \mathbf{E}}{\partial t^2}\right)} \qquad (125)$$

This is the three-dimensional *wave equation* for **E**, the electric field. In fact it represents three separate equations for each cartesian component of **E**, remembering the discussion of the Laplacian in Section 3.3. An analogous equation exists for the magnetic field **B**.

Plane waves Let's consider propagation along $\hat{\mathbf{z}}$, and **polarization**, defined as *the direction of* **E**, along $\hat{\mathbf{x}}$ (vertical). Taking the Laplacian gives us

$$\nabla^2\mathbf{E} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}\hat{\mathbf{x}} \qquad (126)$$

all other cartesian components being zero. Since the propagation direction is along $\hat{\mathbf{z}}$, there is no variation over the other terms and we have

$$\nabla^2 \mathbf{E} = \frac{\partial^2 E_x}{\partial x^2}\hat{\mathbf{x}} \tag{127}$$

$$\left(\frac{\partial^2 E_z}{\partial z^2}\right) = \mu_0\epsilon_0 \left(\frac{\partial^2 E_x}{\partial t^2}\right) \tag{128}$$

One can immediately recognize this expression as the wave equation for EM waves if the velocity v is given by

$$\frac{1}{v^2} = \mu_0\epsilon_0 \tag{129}$$

and indeed it is:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{130}$$

is the speed of light, 3.00×10^8 m/s. The equation can be solved by the complex form

$$\widetilde{\mathbf{E}}(z,t) = \widetilde{A}e^{i(kz-\omega t)}\hat{\mathbf{x}} \tag{131}$$

6 The electron swarm (AC): metallic reflectivity

In this section, we'll see the effect of free electrons on the optical properties of materials (metals and semiconductors). Recall Maxwell's equations in materials (SI units):

$$\nabla \cdot \mathbf{B} = 0 \tag{132}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \tag{133}$$

$$\nabla \times \mathbf{B} = \mu\mathbf{J} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t} \tag{134}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{135}$$

where μ is the (magnetic) permeability, and ϵ is the (dielectric) permittivity. The last two equations are those responsible for electromagnetic wave propagation in solids. A propagating wave takes the form

$$A(\mathbf{r},t) = \mathbf{A}_0 \exp(-i(\mathbf{k}\cdot\mathbf{r} - \omega t)) \tag{136}$$